Nautical Astronomy : From the Sailings to Lunar Distances

Joel Silverberg
Dept of Mathematics
Roger Williams University
Bristol, Rhode Island
USA

Presented at York University, Toronto, Ontario, May 26, 2006
Canadian Society for the History and Philosophy of Mathematics

Abstract

From the mid sixteenth century through the mid eighteenth century advances in science, mathematics, and technology enabled the navigator, cartographer, or surveyor to determine both his latitude and longitude from celestial observations. This paper explores the history of the development of those techniques. The period during which these ideas grew from theoretical speculations to practical tools, a period of two and one-half centuries, was spurred on by the financial encouragement of governments and commercial interests and the contributions of major figures such as Kepler, Newton, Briggs, Napier, Vernier, Harrison, Mayer, Flamsteed, Maskelyne, Bowditch, and others.

À partir du mi seizième siècle par le mi dix-huitième siècle, les avances en science, les mathématiques, et la technologie ont permis au navigateur, au cartographe, ou à l'arpenteur de déterminer sa latitude et longitude des observations célestes. Cet article explore l'histoire du développement de ces techniques. La période l'où ces idées se sont développées des spéculations théoriques aux outils pratiques, une période d'environ deux cents et cinquante ans, a été incitée par l'encouragement financier des gouvernements et des interests commerciaux et les contributions des figures principales telles que Kepler, Newton, Briggs, Napier, Vernier, Harrison, Mayer, Flamsteed, Maskelyne, Bowditch, et d'autres.

Introduction

The growing importance of the exploration of the New World spurred an urgent need for enhanced navigational methods beyond dead reckoning and observations of the highest elevation of the sun above the horizon. Although theoretical speculations on how longitude might be determined from celestial observations can be found as early as 1514, it was in the early 1600's that the search began in earnest. The hundred and fifty years between the early seventeenth and the mid eighteenth centuries witnessed the invention of accurate quadrants and sextants; the first published tables of lunar positions with respect to the sun, planets, and fixed stars; the invention of logarithms and logarithmic scales and tables; and the invention of accurate pendulum clocks for land use and chronometers for marine use. This paper will outline these advances and will explore the origins and development of the mathematics that made these methods practical.

Early Navigational Techniques

The earliest navigational techniques were based on sailing within sight of land, hugging the coast from one port to another. As commerce took sailors further afield, the need arose for navigation techniques which could be used out of sight of land. The discovery and use of the magnetic compass became the foundation for these early methods. The earliest European guides to navigation were called portolans and gave compass headings and distances between various ports of call. Some gave detailed descriptions of harbors including descriptions of prominent landmarks and warning of hazards to navigation. More elaborate portolans included sketches of the landscape in various ports to aid the sailor. Some very early examples of portolans [*circa* 1400] can be viewed at the Dibner Institute's web site devoted to the manuscript of Michael of Rhodes. [1]

Parallel Sailing

When sailing long distances out of sight of land, as was necessarily the case for transatlantic travel (and for some Mediterranean routes of travel), the practice was at first to sail as closely as possible along a parallel of latitude. This procedure was used because it is possible to verify the predicted latitude. At dawn, noon, and dusk, the latitude provided by the dead reckoning was compared with the observed latitude and changed if necessary. But no procedure was available to validate the longitude of the dead reckoning. Sailors developed the strategy of sailing northward or southward until they reached the desired latitude, validated by measuring the altitude of the sun at noon or the altitude of the pole star at dawn or dusk, then attempting to sail eastward or westward along that

[1] See http://dibinst.mit.edu/michaelofrhodes/manuscript/.

parallel until the ship approach a coastline. If the ship missed the desired port, another noon observation could tell them whether to travel northward along the coast or southward until they reached their destination.

Other Sailings

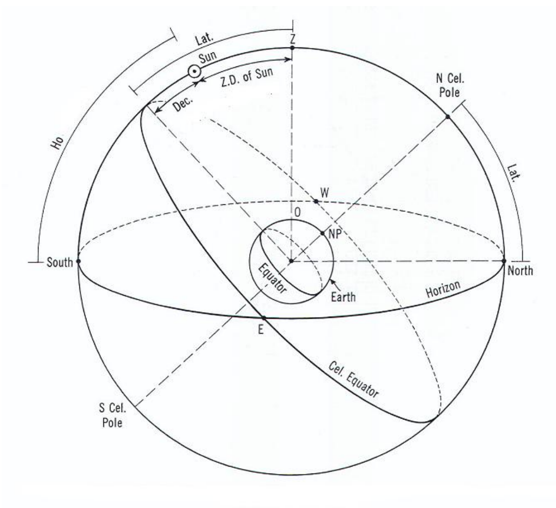
But the helmsman can rarely choose his direction of sail. The direction in which a ship can travel is greatly constrained by the direction of the wind. The path of ships is also effected by unseen currents and the tendency of the vessel to slip to leeward (downwind). The method of parallel sailing was a hazardous one as well, and many a ship was lost on a reef or shoal before it reached the coast if it sailed into waters other than those the navigators thought they were approaching.

Methods which allowed ships to take their natural course with respect to the wind, rather than attempting to sail along a parallel of latitude were used in some parts of the Mediterranean as early as 1400 [**?**] and were refined in the 16th and 17th century following the insights of Gerard Mercator (ca. 1556) and Edward Wright (ca. 1596). [**?**, **?**, **?**, **?**] With the development of nautical charts uthe Mercator projection, and the development of tables of "meridional parts," sailors adapted these techniques to take into account that ships sailed, not upon a plane surface, but upon the surface of a globe, and that they sailed neither upon straight lines nor upon great circle routes, but upon loxodromic spirals. The complex set of methods that produced a correct dead reckoning for a ship sailing upon a series of loxdromes was known as "the sailings." The methods of the various sailings, used for transatlantic navigation until well into the 19th century, are extensively discussed in a previous article prepared for this journal. [**?**].

The Determination of Latitude: The Noon Sight

Perhaps the most fundamental of celestial observations was the noon sight. The sun is at its highest elevation above the horizon at local apparent noon. The sextant (and before that the backstaff, cross staff, or astrolabe) was used to measure the altitude of the sun above the horizon for a period of time bracketing local noon. The maximum altitude could be used to find the latitude of the observer and the moment when that maximum occurred was used to mark the moment of local noon. The angular distance of the sun above or below the celestial equator varies with the seasons throughout the year and was available in tables of "declinations" for each day of the year from classical times.

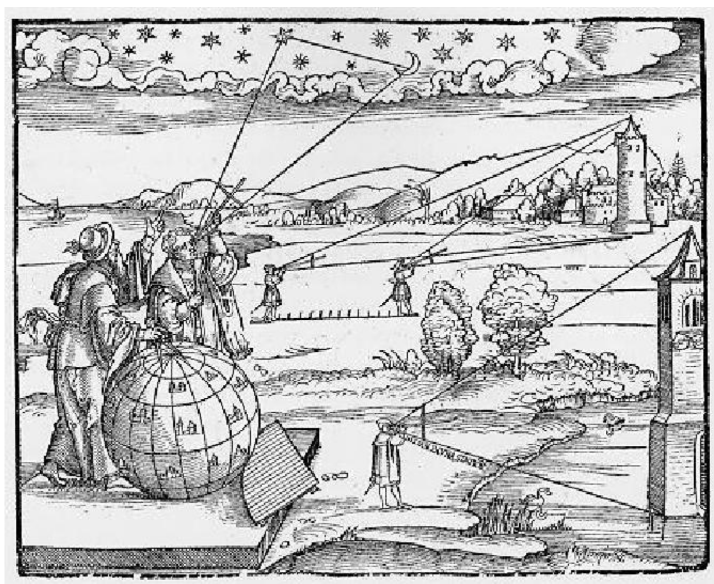
At local noon, the sun, the zenith (the point in the sky directly overhead of the observer), and the celestial poles all lie on the same meridian. The latitude is the arc length from the celestial equator to the zenith. If, on a given day, the position of the sun is north of the celestial equator, as illustrated in the diagram below, the latitude of the observer (arc length of the observer's meridian, between the equator and the observer) can be found by *either* the arc length of the meridian between the celestial pole and the observer's horizon *or* the arc length of the meridian between the zenith and the celestial equator, each of the arcs being complementary to the same arc.



A twilight observation of the altitude of the pole star above the horizon uses the first possibility, while the zenith distance of the sun (the complement of the altitude above the horizon), corrected for the declination of the sun above or below the celestial equator uses the second possibility. Thus the sailor measured the altitude of the sun above the horizon at its highest point, then subtracted that value from 90° to find the zenith distance of the sun. Then the sailor consulted an almanac to determine the sun's declination above or below the celestial equator for that date. If the sun was above the equator, the declination was added to the zenith distance to determine the observer's latitude, else it was subtracted.

The Determination of Longitude

The early years of the 16th century saw the first theoretical speculations on how one might determine the longitude of a particular location. Johann Werner, a follower of Regiomontanus from Nuremburg produced a work on geography entitled *In Hoc Opere Haec Continentur Moua Translatio Primi Libri Geographicae Cl'Ptolomaei,* published in 1514, in which he describes a method of lunar distances, mapping the positions of fixed stars and calculating the longitude from the passage of the moon across each one. He also describes an instrument with an angular scale on a staff from which degrees could be read off. He makes a study of map projections and gives a method to determine longitude based on eclipses of the moon. In 1531, Gemma Frisius, a Flemish cartographer and astronomer, published *Principiis astronomiae* in which he describes how a clock could be set on departure, keeping absolutely accurate time, and compared with local time on arrival (as determined by solar observation). Both men were important influences on Gerard Mercator who first developed and published maps and charts uwhat has become known as the Mercator projection. Neither Werner's nor Frisius' proposed methods were practical for centuries to come. The difficulties to be faced and overcome were considerable and spread over a wide range of disciplines.



This illustration shows a surveyor ua cross staff to determine the height of a tower, a pair of surveyors ucross staffs to determine how far away a fortification is from their location, and an astronomer (note the globe, quadrant, and dividers) measuring the distance between the moon and either a planet or a bright star near the ecliptic in order to determine his longitude. The illustration appears on the fronticepiece of *Introductio geographica* by Petrus Apianus, published in Ingolstadt in 1533.

In this paper we will concentrate on describing the evolution of the method of lunar distances, as proposed by Johann Werner, although we will also comment from time to time about the similarities and differences between this method and the use of time pieces to determine longitude. We will describe both the observational and computational methods that were developed to determine longitude from lunar distances along with the challenges and difficulties that presented themselves along the way.

Developing Tables of Lunar Positions : Problems in celestial mechanics

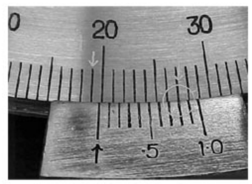
The essence of the lunar distance method is that a table has been prepared that records the position of the moon, sun, and planets against the fixed stars at frequent intervals throughout the day. The sun moves through the fixed stars at a rate of approximately one degree per day. The planets generally move even more slowly, but the moon moves through half a degree of arc in a hour and thus provides a time keeper useful for measuring hours and minutes, rather than days. A detailed and accurate prediction of the moon's orbit is needed to provide such a table. As the contributions of Galileo, Copernicus, and Kepler were applied to this problem, predictions were made, but proved inaccurate when compared to direct observations of the moon's position. Newton also tackled this problem, but in the end accurate predictions of lunar position evaded theoretical solution due to the fact that the moon is strongly influenced by both earth and the sun, a classic "three body problem." An empirical approach was eventually successful. The founding of Greenwich Observatory (and others) and the collection of nearly a century of data culminated in tables of accurate lunar positions. The observatory was established in 1675 and Flamsteed began to collect his lunar observations, a project that was to continue for nearly 40 years.

The first accurate tables of lunar positions were prepared by Tobias Mayer, a professor of economics and mathematics at Göttingen, in 1753 who sent them to the British government in 1755. These tables were good enough to determine longitude at sea with an accuracy of half a degree. It was Mayer who discorvered the libration of the moon and the first edition of Maskelyne's *Nautical Almanac* (1767) included Mayer's tables, with a statement that they were sufficiently accurate to determine one's longitude to within a degree. Comment is also made that the difficulty and length of the necessary calculations seemed the only obstacles to hinder them from becoming of general use, a consideration that we will return to at a later point. Mayer's method of determining longitude by lunar distances and a formula for correcting errors in longitude due to atmospheric refraction were published in 1770 after his death. [**?**]

Accurate tables are a necessary tool for determining longitude, but not sufficient. One also needs to be able to measure both angles and time with great accuracy. The first issue is a technological one, but issues surrounding the measurement of time has, in addition to technical issues, both theoretical and mathematical components.

The problem of angles

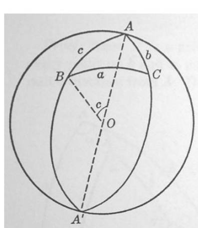
Werner envisioned ua cross staff or an angular scale on a staff to measure the angles required, but it's accuracy fell far short of what was needed. The backstaff was invented by John Davis, an English captain in 1607. The accuracy and usefulness of the medieval quadrant was also improved and we have the first report of a trial of the method of lunar distances at sea in 1615. The sextant was developed shortly afterwards, and soon supplemented by Vernier scales by 1638. By the 1730's the design of sextants and quadrants were refined to their modern form and were capable of measuring angles large and small, and with a precision of a tenth of a degree.

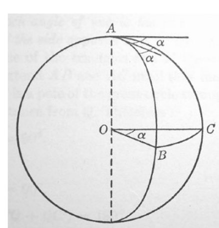


The Celestial Sphere and Spherical Trigonometry

The earth and the observer are imagined to sit at the center of a large sphere, upon which are projected the positions of the sun, moon, planets, and stars. Also projected upon this sphere are the equator and poles of the earth, the lines of latitude and longitude on the earth's surface, and the horizon and zenith as seen by the observer.

Spherical triangles are formed on the surface of a sphere by the intersection of three great circles. In spherical trigonometry both the sides and vertices of a spherical triangle are measured in degrees of arc. The length of a side is measured by the central angle defining the portion of the great circle that is the side of the triangle. In the figure left hand below, the central angle c is the measure of side c on the surface of the sphere.





The vertex angles are measured by the dihedral angle of the planes defining the great circles that form the two sides meeting at that vertex. In the right hand figure above, the vertex labeled A is formed by the intersection of two planes pasthrough the center of the sphere. The angle between these two planes at the center of the sphere, labeled α is the measure of the vertex angle at A.

Each spherical triangle has three sides and three vertex angles, each measured in degrees or minutes of arc. In the figure shown below, the measure of the three sides are labeled $a, b,$ and $, c$ and the measure of the three angles are labeled $\alpha, \beta,$ and γ. Given any three of these six quantities, it is possible to determine the remaining three quantities. There is a spherical law of , which looks much like the law of in plane trigonometry, but it is the laws of cothat are most useful for navigational problems. There are two forms of these laws, one which was called the law of cofor sides and one which was called the law of cofor angles.

Lunar Distances: the simple version

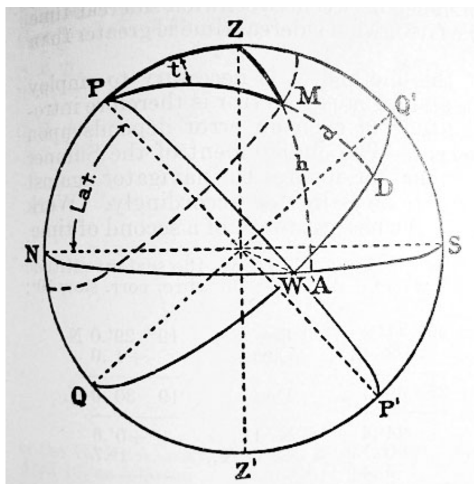
The basic concept of determining lunar distances is to measure the angular separation between the moon and another object (usually the sun) and to find from tables in the Nautical Almanac the moment (Greenwich apparent time) when that angular separation was predicted to occur. Then one calculates the difference between the local apparent time and Greenwich apparent time, and converts the time difference to a difference in longitude (15 degrees of longitude = 1 hour time difference). The implementation is far more complex. The first issue is to consider how to determine the local apparent time. The second issue is that the observed distance between the moon and the sun requires a host of corrections. We consider each in turn.

The problem of local time

It is a misconception that the invention of the marine chronometer removed the need for celestial observations for determining longitude. The chronometer, when working correctly, reports the time at the reference meridian. But the ship's clock can not report the local time – no clockwork, however accurate can determine local time, that time depends upon the observer's position, which on a ship is constantly changing. Only locations on the same meridian share the same local time. Our modern "standard time zones" are more than 1000 miles wide, and all who live within a time zone share the same time, but two locations 50 or 100 miles apart have very different local apparent times. ships are constantly in motion, they are constantly shifting their local time. The noon sight, discussed earlier, provides the sailor with the ship's latitude and the time of noon, and would appear to solve the problem of local time, provided any reasonable timepiece is aboard ship. However the moment at which the sun is at its highest is impossible to measure with accuracy, the sun is at the peak of its arc and its altitude changes very slowly at that point. The sun seems to hang at the same altitude for fifteen to thirty minutes, the uncertainty in the moment when it was exactly noon is considerable. It is when the sun is in the East or in the West that it's altitude above the horizon changes most rapidly, and thus it is in the early morning and late afternoon that the sun's altitude provides the most accurate information about local time, rather than at mid-day when the sun's altitude changes most slowly. The "time sight" was developed to provide an accurate measure of the local time, that is the number of hours before or after local apparent noon.

The basis of the time sight is the spherical triangle formed by the sun, the zenith, and the celestial pole. The three sides of this spherical triangle are the *complementary angles* of the sun's altitude above the horizon, the solar declination , and the observer's latitude.

In the figure below, P marks the celestial north pole, Z the zenith (the point directly above the observer) , and M the position of the sun. The arc NWS marks the observer's horizon, and the arc QWQ' marks the celestial equator. The altitude of the sun above the horizon is labeled h, the declination of the sun above the celestial equator is labeled d, and the latitude is shown as the altitude of the celestial pole above the horizon (rather than the distance from the celestial equator to the zenith). With respect to the spherical triangle PZM, side PZ is a portion of the meridian of the observer, while side PM is a portion of the meridian of the sun. The angle at P, labled t in the figure is therefore a measure of the time that has passed local noon.



The solar altitude is measured ua sextant or similar instrument, the solar declination is determined from the almanac entry for the date of the observation, and the latitude is estimated from a deduced reckoning based on the most recent noon sight and the information on course and speed in the ship's log (as used in the sailings) [**?**]. The ship's latitude can not be determined from direct observation at the time of the "time sight" uthese methods. three sides of a spherical triangle are now known, the three vertex angles can be determined. The angle at the celestial pole is called the local hour angle and is directly proportional to the time , or before, noon. The angle at the zenith is called the azimuth and the angle at the solar position is called the parallactic angle or the angle of position.

Clearing the lunar distance

The time sight provided the sailor with a knowledge of local apparent time. The time sight required for its calculation a previous noon sight (for finding the ship's latitude) and a dead reckoning of the ship's movements the noon sight was performed (so that the ship's latitude could be updated for the position of the ship when the time sight was performed.) The lunar distance was designed to provide the apparent time at the reference meridian, so that the difference in longitude between that meridian and the ship's position could be determined.

Calculating the distance from center to center

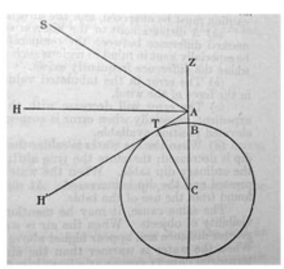
The distance (in degrees of arc) between the moon and the sun was measured, but that distance was measured from limb to limb, that is from the edges of each disk, not from their centers, while the entries in the Nautical Almanac were calculated from the center of the sun's disk to the center of the moon's disk. Thus the first corrections in the measured values were to look up the semi-diameters of the sun and the moon. These vary with the distance of the earth from the sun and of the earth from the moon, so they were tabulated by date.

Augmentation

When the moon is high in the sky, the observer is closer to the moon than he would be if the moon were on the horizon. If the moon were directly overhead, it would be closer by the radius of the earth (about 4,000 miles closer). Thus the higher in the sky the moon is, the larger it's semidiameter. Tables in the almanac provide a correction, called augmentation, for this effect.

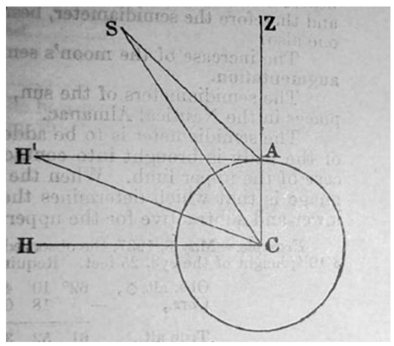
Allowance for dip

The height of the eye above the surface of the water effects the location of the sea horizon. Due to the curvature of the earth, any additional height allows the observer to see further over the curve of the ocean surface. Tables in the almanac provide a correction in altitude to provide the angle that would be observed from the surface of the sea.

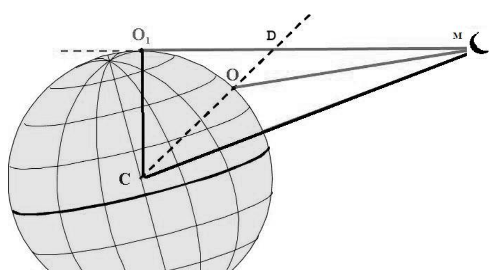


Allowance for parallax

The positions of the sun and moon in the Nautical Almanac are calculated on a geocentric basis ... as seen from the center of the earth. The observer on the surface sees the moon in a different position due to the parallax of the two different locations. the moon is relatively close to the earth, the parallax can be large (nearly one degree at times) and requires correction. In the figure below, the the altitude of the moon, as seen from position A is angle H' A S, while the altitude as "seen" from the center of the earth (as reported in the Almanac) would be angle HCS. A celestial object at point S would appear higher in the sky from the center of the earth at C than it would appear to the observer on the surface at point A.



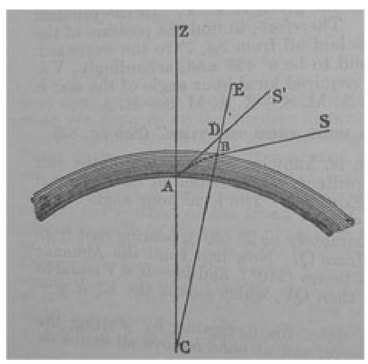
Consider the diagram below. The moon is centered at M, the earth at C, and an observer at point O views the moon at high in the sky. The zenith distance of the moon, as observed from point O is given by angle DOM. An observer at point O_1, however observes the moon to be on the horizon (with a zenith distance of 90 degrees). The angle O_1MC is called the horizontal parallax and is tabulated in the Nautical Almanac. The angle OMC is the parallax for the observer at point O and needs to be computed.



Distance MC is the distance between the earth and the moon, and distances OC and O_1C are equal to each other and are equal to the radius of the earth. Considering triangle O_1MC we see that $\sin(HP) = \frac{r}{MC}$ where HP is the horizontal parallax and r is the radius of the earth. Considering triangle OCM and applying the law of to that triangle, we see that $\frac{\sin(p)}{r} = \sin(180^\circ - zdm)MC$ where p is the local parallax (angle OMC), r is the radius of the earth, and zdm is the zenith distance of the moon. the of the supplement of an angle is equal the the of the angle, we can replace $\sin(180^\circ - zdm)$ by $\sin(zdm)$. Thus $\frac{\sin(p)}{r} = \frac{\sin(zdm)}{MC}$. Solving for $\sin(p)$ we get $\sin(p) = \sin(zdm)\frac{r}{MC} = \sin(zdm)\sin(HP)$. both p and HP are small (1 degree or less), a small angle approximation $\sin(x) \approx x$ can be used to obtain the relationship $p = \sin(zdm)HP$.

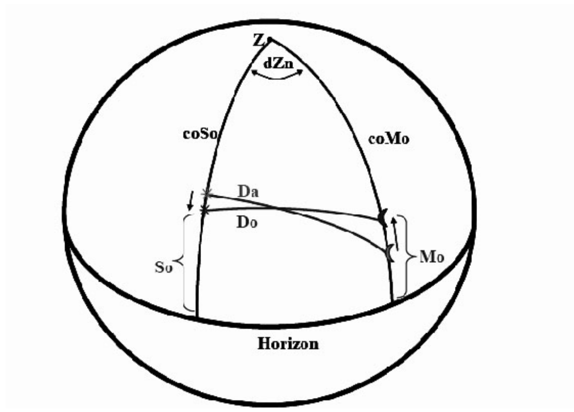
Allowance for refraction

Rays of light traveling through the atmosphere at an angle will be bent through the effects of refraction, in accordance with Snell's Law, so that they appear higher in the sky than they really are. This effect is proportional to the tangent of the zenith angle. Objects at the zenith are not effected, objects near the horizon experience the greatest effects. Tables for the effects of refraction appear in the nautical almanacs. Nineteenth and twentieth century almanacs may also have corrections for temperature and barometric pressure, which effect the density of the atmospheric gases.



Computational difficulties in clearing the lunar distance

The effects of dip, parallax, augmentation, and refraction all effect the altitude at which the moon, sun, and stars are observed, but not their horizontal position in the sky. With the aid of modern computers and calculators, a straight forward way of determining their effect on the lunar distance can be formulated. [?]



First, measure the lunar distance, correcting for semidiameter and augmentation. Next, measure the altitude of sun and moon and calculate their complements. Thirdly, use the spherical law of cofor sides to calculate the vertex angle at the zenith. Adjust the altitudes for sun and moon for the effects of dip, parallax, and refraction. Recalculate complements of the adjusted altitudes, and use the zenith vertex angle found with the uncorrected sides. Note that the zenith vertex angle is not effected by changes in altitudes. Uthe two sides (complements of corrected altitudes) and the vertex angle, use the spherical law of cofor angles to calculate the length of the side connecting sun and moon. That is your corrected lunar distance. Consult the tables in the Nautical Almanac to find out what time it was in Greenwich when the bodies were so separated. That is apparent time in Greenwich. Find the difference between that time your local apparent time as determined by your time sight, and convert the time difference to a longitudinal difference. This is an elegant solution, but the computational burdens of carrying this out in the absence of electronic calculators and computers were insurmountable. Another method needed to be found. Two major mathematical contributions enabled the lunar distances to be computer without the aid of electronic computers (although many human computers were in fact used in preparing the tables for these almanacs). The first of these contributions were the invention of logarithms by Napier and their perfection by Briggs. The second of these resulted from increasophisticated sets of trigonometric manipulations, culminating, in clodecades of the eighteenth century, in scores of competing methods, direct and approximate, for clearing the lunar distance.

The problem of computational complexity

The central computation for performing a time sight is to solve the equation

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha) \tag{1}$$

for the unknown hour angle α, given the sides $a, b,$ and c. Thus

$$\cos(\alpha) = \frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)} \tag{2}$$

or

$$\alpha = arc\cos(\frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)}. \tag{3}$$

Tables of and coof sufficient accuracy were available from the mid sixteenth century, but the multiplication of two 6 or 7 digit numbers was both tedious and error prone, the division of two such numbers even more so. The discovery of logarithmic numbers and the prompt development of tables of logarithms and logarithms of trigonometric functions at about the same time, eased such problems of multiplication and division, but the form of equation (**??**) or equation (**??**) does not lend itself to logarithmic computation due to the subtraction in the numerator.

Adaptations for Logarithmic Computation

many of the trigonometric equations that navigation required involved finding sums and differences, as well as products and quotients, it was often the case that logarithmic computations were not as useful as one would wish. Considerable ingenuity was exercised to find trigonometric identities and manipulations that would change the needed calculations into the form of products and quotients. Often this entailed employing trigonometric functions less common than chords, , tangents, and secants. Occasionally it entailed inventing new trigonometric functions, and of course computing and making available trigonometric and log-trigonometric tables for these unusual or new functions. Tables of the versed , haver, suver, and log squares, were a part of many sailor's toolkit. [**?**]

These functions often have interesting geometric interpretations in terms of the chords, tangents, and secants of the circles through which such functions were understood before the late eighteenth century, and in addition they often simplified the equation into a form which could be more easily memorized and remembered. The versed was understood as that part of the diameter between the and the arc, the haverwas defined as one-half of the versed , the coverwas the versed of the complement of the arc, and the suveras the difference between the diameter and the versed . [2]

[2]Algebraically, $vers(x) = 1 - \cos(x)$, $hav(x) = \frac{1}{2}vers(x)$, $suvers(x) = 1 + \cos(x)$, and $covers(x) = 1 - \sin(x)$. The functions are also closely linked with the half-angle identities. $\sin^2(\frac{x}{2}) = hav(x)$ and $\cos^2(\frac{x}{2}) = suvers(x)$.

The Versed

The versed was used occasionally in navigational computations throughout the 18th century, but became more widely appreciated in the 19th century as use of lunar distances to determine longitude required more intricate calculations. The use of the versed avoided the potential errors caused by the algebraic sign of the co, tangent, or secant functions of second quadrant angles. The function results in ambiguous cases where two different directions have the same . The cofunction requires the consistent and accurate use of signed arithmetic or algebraic sums. The versed however has neither problem, but increases monotonically from 0 to 2 as the angle passes through the first two quadrants.

The Haver

The haveris defined as one-half of the versed , and arises naturally as a part of the half-angle identities. Today we teach students that $\sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{2}}$, but until the nineteenth century this would have been understood as $\sin^2(\frac{x}{2}) = hav(x)$. This formula is much easier to remember and is very suitable for logarithmic calculation.

Recall the equation for finding the local hour angle in a time sight:

$$\cos(\alpha) = \frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)}. \tag{4}$$

A common eighteenth century manipulation was to introduce the variable $s = \frac{a+b+c}{2}$, called the half-sum. Uequation (**??**),

$$\begin{aligned} vers(\alpha) = 1 - \cos(\alpha) = 1 - \frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)} \\ = \frac{\sin(b)\sin(c) + \cos(b)\cos(c) - \cos(a)}{\sin(b)\sin(c)} \\ = \frac{\cos(b-c) - \cos(a)}{\sin(b)\sin(c)} \end{aligned} \tag{5}$$

Adding the expansions $\cos(A-B)$ and $\cos(A+B)$leads to the identity $\cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B)$ thus replacing a difference by a product. The offending difference in our problem is $\cos(a) - \cos(b-c)$ Setting $A+B = a$ and $A-B = b-c$ we find that $A = \frac{a+b-c}{2} = (s-c)$ and $B = \frac{a-b+c}{2} = (s-b)$. Thus the expression $\cos(b-c) - \cos(a)$ can thus be replaced by $2\sin(\frac{a-b+c}{2})\sin(\frac{a+b-c}{2})$ and equation (**??**) becomes

$$vers(\alpha) = 2\frac{\sin(s-b)\sin(s-c)}{\sin(b)\sin(c)} \tag{6}$$

. In terms of haveor square ,

$$hav(\alpha) = \sin^2(\frac{\alpha}{2}) = \frac{\sin(s-b)\sin(s-c)}{\sin(b)\sin(c)} \tag{7}$$

or even more concisely,

$$hav(\alpha) = csc(b)csc(c)\sin(s-b)\sin(s-c) \tag{8}$$

For a time sight, $a, b,$ and c are the complement of the altitude, $90° - h$, the polar distance, p, and the complement of the latitude, $90° - \phi$, respectively. In terms of these variables $\cos(s-b) = \cos(90° - \frac{h+p+\phi}{2})$ and $\cos(s-c) = \cos(\frac{p+\phi+h}{2} - h)$.

If α represents the local hour angle, the equation becomes

$$hav(\alpha) = csc(p)sec(\phi)\cos\left(\frac{h+p+\phi}{2}\right)\cos\left(\frac{h+p+\phi}{2} - h\right) \tag{9}$$

Note how perfectly suited for logarithmic calculations this formulation of the problem is. The items involved in addition and subtraction are direct measurement, not the results of intermediate calculations. Furthermore, whereas the intermediate calculations (based on looking values up in logarithmic or trigonometric tables) required 6 to 8 significant figures of accuracy, the direct measurements required many fewer significant figures. Once the table lookups are performed, determing the value of $loghaver(\alpha)$ required only sums and differences, along with one more table lookup to obtain the inverse log haver. Many other such rearrangements appear in the manuals for seamen and the textbooks for navigation. Hardly ever are explanations included for where these remarkable formulas came from.

The expression $\frac{h+p+\phi}{2}$ was called the half-sum, and the expression $\frac{h+p+\phi}{2} - h$ was called the remainder. Such forms populate many pages of Bowditch's manual for seamen and are reflected in the training of seamen up into the twentieth century. An oral history of one such sailor recorded this conversation how he was taught to find local time.

"Add the secant of latitude, the of polar distance, the coof the half sum, and the of the remainder ... the logs of course... and you gotter remember what it gives yer ... it gives yer the log haverof the hour angle" [?]

This semi-poem exactly reflects equation (**??**), and although it is doubtful that this elderly seaman knew where such a rule came from, he knew exactly what he was doing when he used it.

From Mendoza y Rios to Bowditch

Don José de Mendoza y Rios was a retired Spanish admiral, an expatriate during the wars between England and Spain, and a member of the Royal Society of London. He presented a monumental overview of nautical astronomy and navigational science to the Society in 1796. As published in the Philosophical Transactions of the society it ran to seventy-nine pages of detailed explanations and complex trigonometric calculations. Written in the French language, it was "read" to the Society by Sir Joseph Banks, and detailed no fewer than forty different methods for clearing the lunar distance. I have selected one to present here both because the method is interesting and because it was very widely used by navigators in the late 18th and early 19th century. With some adjustments it becomes incorporated in Nathaniel Bowditch's ground breaking work, the New American Practical Navigator (1802).

The goal of clearing a lunar distance was to correct all apparent measurements to "true" ones, that is, measurements as they would be seen by a hypothetical observer at the center of the earth without the distorting effects of the earth's atmosphere. The corrected value of the lunar distance is then compared with the table of lunar distances in the Nautical Almanac, and interpolating between the table entries (recorded at three hour intervals), determining the apparent time at the reference meridian.

Let M and S be the apparent lunar and solar altitudes respectively, and let m and s be the true lunar and solar altitudes. Let D be the apparent lunar distance and d be the true lunar distance and let Z be the zenith angle.

Then the spherical triangle MZS leads to the relationship

$$\cos(D) = \cos(Z)\cos(M)\cos(S) + \sin(M)\sin(S) \tag{10}$$

via the spherical law of cofor sides, while the triangle mZs leads to the equation

$$\cos(d) = \cos(Z)\cos(m)\cos(s) + \sin(m)\sin(s). \tag{11}$$

Solving equation(**??**) for $\cos(Z)$ we get

$$\cos(Z) = \frac{\cos(d) - \sin(m)\sin(s)}{\cos(m)\cos(s)}. \tag{12}$$

Substituting this value for $\cos(Z)$ into equation(**??**) we get

$$\cos(D) = \frac{\cos(d) - \sin(m)\sin(s)}{\cos(m)\cos(s)}\cos(M)\cos(S) + \sin(M)\sin(S)$$

$$\cos(D) = (\cos(d) - \sin(m)\sin(s))\frac{\cos(M)\cos(S)}{\cos(m)\cos(s)} + \sin(M)\sin(S)$$

$$\cos(D) = \frac{\cos(d)\cos(M)\cos(S)}{\cos(m)\cos(s)} - tan(m)tan(s)\cos(M)\cos(S) + \sin(M)\sin(S) \tag{13}$$

The moon appears lower in the sky than its geocentric position would indicate, $M = m + u$ where u is positive. The effect of refraction on the moon is much less than the effect of parallax. The sun appears higher in the sky that its true postion (due to refraction) $S = s - v$, where v is positive. The parallax of the sun is negligible in comparison with the effect of refraction the sun is so far away from the earth.

Expanding the and coof the sum and differences of angles ubasic trigonometric identies we see that

$$\sin(M) = \sin(m + u) = \sin(m)\cos(u) + \cos(m)\sin(u) \tag{14}$$

$$\cos(M) = \cos(m + u) = \cos(m)\cos(u) - \sin(m)\sin(u) \tag{15}$$

$$\sin(S) = \sin(s - v) = \sin(s)\cos(v) - \cos(s)\sin(v) \tag{16}$$

$$\cos(S) = \cos(s - v) = \cos(s)\cos(v) + \sin(s)\sin(v). \tag{17}$$

Uequations (**??**) through (**??**) to calculate $\sin(M)\sin(S)$ and $\cos(M)\cos(S)$, which we will need for equation (**??**) we get

$$\begin{aligned}\sin(M)\sin(S) = {} & \sin(m)\sin(s)\cos(u)\cos(v) + \cos(m)\sin(s)\sin(u)\cos(v) \\ & - \sin(m)\cos(s)\cos(u)\sin(v) - \cos(m)\cos(s)\sin(u)\sin(v)\end{aligned} \tag{18}$$

and

$$\begin{aligned}\cos(M)\cos(S) = {} & \cos(m)\cos(s)\cos(u)\cos(v) - \sin(m)\cos(s)\sin(u)\cos(v) \\ & + \cos(m)\sin(s)\cos(u)\sin(v) - \sin(m)\sin(s)\sin(u)\sin(v)\end{aligned} \tag{19}$$

Medoza y Rios expands $\sin(u), \sin(v), \cos(u), and \cos(v) intoaTaylorseriesexpansion, keepingalltermsofordert$
$u, \sin(v) \approx v, \cos(u) \approx 1 - \frac{u^2}{2}$, and $\cos(v) \approx 1 - \frac{v^2}{2}$. Substituting the values u and v for $\sin(u)$ and $\sin(v)$ and substituting $1 - \frac{u^2}{2}$ for $\cos(u)$ and $1 - \frac{v^2}{2}$ for $\cos(v)$ in equations (**??**) and (**??**) we can express $\sin(M)\sin(S)$ and $\cos(M)\cos(S)$ in terms of m, s, u,and v.

$$\begin{aligned}\sin(M)\sin(S) = {} & \sin(m)\sin(s) + u\ \cos(m)\sin(s) - v\ \sin(m)\cos(s) \\ & - u\ v\ \cos(m)\cos(S) - \frac{1}{2}u^2\ \sin(m)\sin(s) - \frac{1}{2}v^2\ \sin(m)\sin(s)\end{aligned} \tag{20}$$

$$\begin{aligned}\cos(M)\cos(S) = {} & \cos(m)\cos(s) - u\ \sin(m)\sin(s) + v\ \cos(m)\sin(s) \\ & - u\ v\ \sin(m)\sin(s) - \frac{1}{2}u^2\ \cos(m)\cos(s) - \frac{1}{2}v^2\ \cos(m)\cos(s)\end{aligned} \tag{21}$$

Substituting the values from equations (**??**) and (**??**) into the expression for $\cos(D)$ in equation (**??**) we get

$$\cos(D) = \cos(d) + \frac{u\ \sin(s)}{\cos(m)} - u\ \cos(d) tan(m) - \frac{v\ \sin(m)}{\cos(s)} + v\ \cos(d) tan(s)$$
$$+ \frac{u\ v\ (\sin^2(m) - \cos^2(s) - \cos(d)\sin(m)\sin(s))}{\cos(m)\cos(s)} - \frac{1}{2}\ u^2\cos(d) - \frac{1}{2}\ v^2\cos(d). \tag{22}$$

Taking $D = d + \delta$ one has $\cos(D) = \cos(d+\delta) = \cos(d)\cos(\delta) - \sin(d)\sin(\delta)$ and uthe second order approximations for the error term, δ, we can replace $sin(\delta)$ by δ and $\cos(\delta)$ by $1 - \frac{1}{2}\delta^2$ to get

$$\cos(D) = \cos(d) - \delta\sin(d) - \frac{1}{2}\ \delta^2\cos(d) \tag{23}$$

which when substituted into equation(**??**) gives the correction needed for the lunar distance in terms of the apparent lunar distance and the apparent lunar and solar altitudes, along with the corrections to be made in the lunar and solar altitudes:

$$\delta = -\frac{u\ \sin(s)}{\cos(m)\sin(d)} + u\ cot(d) tan(m) + \frac{v\ \sin(m)}{\cos(s)\sin(d)} - v\ cot(d) tan(s)$$
$$+ \frac{u\ v\ (\sin^2(m) - \cos^2(s) + \cos(d)\sin(m)\sin(s))}{\sin(d)\cos(m)\cos(s)} + \frac{1}{2}u^2 cot(d) + \frac{1}{2}v^2 cot(d)$$
$$- \frac{1}{2}\delta^2 cot(d). \tag{24}$$

Note, however, that the desired lunar distance correction, δ appears on *both* sides of this equation. Mendoza takes an additional step, regrouping the terms of the above equation in terms of powers of u, v, and δ.

$$\delta = -u\ \frac{\sin(s) - \cos(d)\sin(m)}{\sin(d)\cos(m)} + v\frac{\sin(m) - \cos(d)\sin(s)}{\sin(d)\cos(s)}$$
$$+ u\ v\ \frac{\cos(d)\sin(m)\sin(s) - \sin^2(m) + \cos^2(s)}{\sin(d)\cos(m)\cos(s)} + \frac{1}{2}u^2 cot(d)$$
$$+ \frac{1}{2}v^2 cot(d) - \frac{1}{2}\delta^2 cot(d) \tag{25}$$

and then procedes to square that above value for δ, eliminating any products of u, v, or δ that are higher than second order. He gets a second order approximation for δ^2

$$\delta^2 = u^2(\frac{\sin(s) - \cos(d)\sin(m)}{\sin(d)\cos(m)})^2 + v^2(\frac{\sin(m) - \cos(d)\sin(s)}{\sin(d)\cos(s)})^2$$
$$- 2\ u\ v\ (\frac{\sin(s) - \cos(d)\sin(m)}{\sin(d)\cos(m)})(\frac{\sin(m) - \cos(d)\sin(s)}{\sin(d)\cos(s)}), \quad (26)$$

which he subsitutes for δ^2 in the right hand side of equation(**??**).

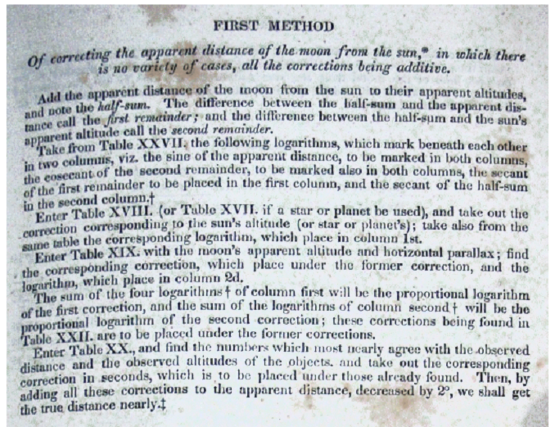
$$\delta = -u(\frac{\sin(s) - \cos(d)\sin(m)}{\sin(d)\cos(m)}) + v(\frac{\sin(m) - \cos(d)\sin(s)}{\sin(d)\cos(s)})$$
$$+ u\ v\ \frac{2\cos(d)\sin(m)\sin(s) + \sin^2 d - \sin^2(m) - \sin^2(s)}{\sin^2(d)\cos(m)\cos(s)}$$
$$+ \frac{1}{2}u^2 cot(d)\left(1 - \left(\frac{\sin(s) - \cos(d)\sin(m)}{\sin(d)\cos(m)}\right)^2\right)$$
$$+ \frac{1}{2}v^2 cot(d)\left(1 - \left(\frac{\sin(m) - \cos(d)\sin(s)}{\sin(d)\cos(s)}\right)^2\right). \quad (27)$$

Medoza concudes, “Voilà la formule qui exprime généralement les corrections qu’on doit appliquer à la distance apparente d, pour avoir la distance vraie D, ayant égard à toutes les équations qui dérivent de u, v, et des produits du second ordre de ces éléments.” [**?**] [3]

Of course, no one aboard ship would be expected to carry out such calculations. As much as possible would be calculated ahead of time, on land, and presented to the sailor as a series of tables, which he would employ to find his various corrections, and these tables were detailed and numerous, forming the vast majority of the bulk of any nautical almanac or manual for seamen.

[3] [And here we have the formula which generally expresses the corrections that one must apply to the apparent [lunar] distance d, to determine the true [lunar] distance D, including in all equations the effects of any terms which involve u and v [the corrections for for lunar and solar altitude] and any second order products of those quantities.]

Days	Stars Names.	12 Hours. ° ′ ″	15 Hours. ° ′ ″	18 Hours. ° ′ ″	21 Hours. ° ′ ″
	[48] APRIL 1767. Distances of ☽'s Center from ☉, and from Stars west of her				
1	The Sun.	40. 59. 11	42. 34. 44	44. 9. 51	45. 44. 35
2		53. 32. 7	55. 4. 24	56. 36. 16	58. 7. 45
3		65. 39. 18	67. 8. 27	68. 37. 14	70. 5. 39
4		77. 22. 36	78. 48. 58	80. 15. 1	81. 40. 46
5		88. 45. 20	90. 9. 27	91. 33. 21	92. 57. 0
6		99. 52. 6	101. 14. 34	102. 36. 52	103. 59. 1
7		110. 47. 42	112. 9. 6	113. 30. 25	114. 51. 40
6	Aldebaran	50. 36. 10	52. 4. 5	53. 31. 57	54. 59. 44
7		62. 17. 43	63. 45. 10	65. 12. 34	66. 39. 57
8	Pollux.	31. 25. 48	32. 53. 11	34. 20. 40	35. 48. 12
9		43. 7. 5	44. 35. 4	46. 3. 8	47. 31. 15
10	Regulus.	17. 51 57	19. 20. 36	20. 49. 26	22. 18. 27
11		29. 45. 36	31. 15. 26	32. 45. 26	34. 15. 35
12		41. 48. 49	43. 19. 55	44. 54. 10	46. 22. 36
13		54. 2. 11	55. 34. 36	57. 7. 12	58. 39. 59
14		66. 26. 28	68. 0. 18	69. 34. 20	71. 8. 33
15	Spica ♍	25. 4. 34	26. 39. 23	28. 14. 26	29. 49. 44
16		37. 49. 37	39. 26. 14	41. 3. 5	42. 40. 8
17		50. 48. 40	52. 26. 59	54. 5. 31	55. 44. 15
18		64. 1. 2	65. 41. 3	67. 21. 18	69. 1. 48
19	Antares.	31. 37. 14	33. 19. 7	35. 1. 13	36. 43. 32
20		45. 18. 29	47. 2. 10	48. 46. 5	50. 30. 12
21		59. 14. 6	60. 59. 31	62. 45. 11	64. 31. 2
22		73. 23. 37	75. 10. 43	76. 58. 2	78. 45. 31
23	α Capricorni.	33. 17. 26	35. 4. 38	36. 52. 4	38. 39. 45
24		47. 41. 9	49. 29. 53	51. 18. 44	53. 7. 40
25	α Aquilæ.	65. 57. 35	67. 29. 54	69. 2. 36	70. 35. 39
26		78. 24. 51	79. 59. 6	81. 33. 29	83. 7 45



FIRST METHOD

Of correcting the apparent distance of the moon from the sun, in which there is no variety of cases, all the corrections being additive.*

Add the apparent distance of the moon from the sun to their apparent altitudes, and note the *half-sum*. The difference between the half-sum and the apparent distance call the *first remainder*; and the difference between the half-sum and the sun's apparent altitude call the *second remainder*.

Take from Table XXVII. the following logarithms, which mark beneath each other in two columns, viz. the sine of the apparent distance, to be marked in both columns, the cosecant of the second remainder, to be marked also in both columns, the secant of the first remainder to be placed in the first column, and the secant of the half-sum in the second column.†

Enter Table XVIII. (or Table XVII. if a star or planet be used), and take out the correction corresponding to the sun's altitude (or star or planet's); take also from the same table the corresponding logarithm, which place in column 1st.

Enter Table XIX. with the moon's apparent altitude and horizontal parallax; find the corresponding correction, which place under the former correction, and the logarithm, which place in column 2d.

The sum of the four logarithms† of column first will be the proportional logarithm of the first correction, and the sum of the logarithms of column second† will be the proportional logarithm of the second correction; these corrections being found in Table XXII. are to be placed under the former corrections.

Enter Table XX., and find the numbers which most nearly agree with the observed distance and the observed altitudes of the objects. and take out the corresponding correction in seconds, which is to be placed under those already found. Then, by adding all these corrections to the apparent distance, decreased by 2°, we shall get the true distance nearly.‡

On the left is a page from the first edition of the Nautical Almanac, prepared by Nevil Maskelyne. On the right is a set of instructions for uthe tables for clearing the lunar distance, from an 1801 American edition of John Hamilton Moore's *New Practical Navigator*.

The mathematical creativity, ingenuity, and sheer tenacity that underlies the calculations that were behind the tables and methods made available in such works is impressive. That being said, it was not expected that the navigator or sailor understood or cared where such methods came from, only that they worked and that they were within his abilities to carry out with accuracy in the difficult envirnoment of a ship at sea. "Seamen of all times have been content to work according to the rule, caring little for the derivation of the rule. ... Moreover, once a specified method ... had been accepted, mastered and committed to memory, a seaman tended to use it throughout his sea-going career." [**?**, p. 246] The same comment would not hold true for the inventors of these methods, their advocates in the admiralty, and the teachers of navigation, who often came to their positions with considerable mathematical talent and continually strove to improve the accuracy and efficacy of these methods.

References

[1] Bok, B.J. and Wright, F.W., *Basic Marine Navigation,*, Houghton Mifflin, Boston, New York, 1944.

[2] Bowditch, Nathaniel, *The New American Practical Navigator,* F. & G. W. Blunt, New York, 1833.

[3] Brink, R.W., *Spherical Trigonometry*, D. Appleton-Century Company, Inc., NY, London, 1942.

[4] Calahan, H. A., *The sky and the sailor; a history of celestial navigation* (1st ed. ed.). Harper, New York, 1952.

[5] Cotter, Charles H., *A History of Nautical Astronomy*, Hollis & Carter, London, 1968.

[6] Ifland, P. *Taking the Stars: Celestial Navigation from Argonauts to Astronauts*, Mariners' Museum, Krieger Pub. C., Newport News, VA, 1998.

[7] O'Connor, J. J. and Robertson, E. F., *Tobias Mayer*, The MacTutor History of Mathematics Archive, University of St. Andrews, Scotland, www-gap.dcs.st-and.ac.uk/˜ history/Biographies/Mayer_ Tobias.html.

[8] Michael of Rhodes: A Medieval Mariner and His Manuscript. Dibner Institute for the History of Science and Technology, http://dibinst.mit.edu/DIBNER/Rhodes/manuscript, 2005.

[9] Petze, C.L., *The Evolution of Celestial Navigation*, Motor Boating, New York, 1948.

[10] Rios, Joseph de Mendoza y , *Recherches sur les principaux Problems de l'Astronomie Nautique.* Par Don Jesef de Mendoza y Rios, F.R.S. Communicated by Sir Joseph Banks, Bart. K.B.P.R.S ., December 22, 1796. Philisophical Transactions of the Royal Society of London, Series I, Vol. 87, pp 43 – 122, 1797.

[11] Silverberg, Joel, *The Sailings: The Mathematics of Eighteenth Century Navigation in the American Colonies,* Proceedings of the Canadian Society for the History and Philosophy of Mathematics, Volume 18, 20005, pp 173–199.

[12] Vincent, Fiona, Positional Astronomy, University of St.Andrews, Scotland, 1998 and 2003. http://star-www.st-and.ac.uk/˜ fv/webnotes/index.html

[13] Wroth, L.C., *Some American Contributions to the Art of Navigation, 1519–1802,*, Associates of the John Carter Brown Library, Providence, 1947.

Appendix A: List of Manuscripts Consulted

Rhode Island Historical Society Manuscripts

- 1712 Jahleel Brenton, age 22, aristocrat, merchant, captain
- 1719 James Browne, age 18, ship owner, merchant
- 1726 Edouard LeGros, age unknown, Newport(?) seaman and merchant
- 1750 Moses Brown, age 12, merchant, industrialist, educator, Quaker, abolitionist
- 1753 John Brown, age 17, merchant, China Trade
- 1763 & 1770 George Arnold, age 16 & 23, captain of both fishing and trading vessels
- 1792 Eliab Wilkinson, age 19, schoolteacher, almanac writer, surveyor, banker
- 1792 George Utter Arnold, age 16, mill owner, store owner, justice of the peace
- 1805–1818 Martin Page, age 15, Seaman, Captain, Ship's Master and supercargo for Brown and Ives, merchants in the West Indies and China Trades
- 1829, 1835, 1840 Viets Peck, age 15 – 26, Merchant, Captain, father involved in slaving and smuggling at Port Royal (Jamaica) & Havana